

# Higher-order mesoscopic fluctuations in quantum wires: Conductance and current cumulants

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We study conductance cumulants  $\langle\langle g^n \rangle\rangle$  and current cumulants  $C_j$  related to heat and electrical transport in coherent mesoscopic quantum wires near the diffusive regime. We consider the asymptotic behavior in the limit where the number of channels and the length of the wire in the units of the mean free path are large but the bare conductance is fixed. A recursion equation unifying the descriptions of the standard and Bogoliubov–de Gennes (BdG) symmetry classes is presented. We give values and come up with a novel scaling form for the higher-order conductance cumulants. In the BdG wires, in the presence of time-reversal symmetry, for the cumulants higher than the second it is found that there may be only contributions which depend nonanalytically on the wire length. This indicates that diagrammatic or semiclassical pictures do not adequately describe higher-order spectral correlations. Moreover, we obtain the weak-localization corrections to  $C_j$  with  $j \leq 10$ .

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## I. INTRODUCTION

In a quantum wire (quasi-one-dimensional geometry), the quantities of interest, such as conductance, are not selfaveraging.<sup>1</sup> Thus, a statistical description of a variety of quantum effects, such as universal conductance fluctuations, requires the conductance cumulants  $\langle\langle g^n \rangle\rangle$  or conductance distribution  $P(g)$  to be studied. Cumulants of current  $C_j$ , like  $C_2$  that is directly related to shot noise, also yield information which is not contained in the averaged conductance. Higher-order mesoscopic fluctuations, though smaller, provide important additional information about the transport process. For example, the recent experiments<sup>2</sup> where an anomalously asymmetric  $P(g)$  was detected have spurred dispute<sup>3</sup> and interest on higher conductance cumulants in mesoscopic wires. This is due to the fact that noninteracting one-parameter scaling models suggest that near the diffusive region the conductance distribution  $P(g)$  is close, but not identical, to Gaussian shape.<sup>4</sup> For the moment, current-cumulant measurements on the third cumulant  $C_3$  have been carried out<sup>5</sup> and a scheme for a detection of  $C_4$  has recently been put forward.<sup>6</sup>

The fundamental symmetries of the Hamiltonian of a disordered conductor manifest themselves in the statistical properties of the energy levels and particle states. The implications of symmetry are conveniently analyzed within the framework of random matrix theory.<sup>7</sup> There are altogether ten different random matrix theories for different symmetric spaces<sup>8</sup> of the Hamiltonian.<sup>9</sup> The symmetry classes are customarily referred by the Cartan's symbol for the symmetric space of the corresponding Hamiltonian.<sup>9</sup> In this paper we consider seven symmetry classes: A(I,II), C(I), and D(III). Disordered normal metals may be studied through standard or Wigner-Dyson (WD) universality classes A(I,II).<sup>10</sup> The BdG classes C(I) and D(III) are appropriate, e.g., for “disorder-facilitated” quasiparticle transport<sup>11</sup> in unconventional superconductors. Compared to the WD en-

sembles, an extra degree of freedom arises for the BdG classes, since a distinction between particlelike and holelike quasiparticles of the BdG formalism is made. Further, the systems are classified according to the presence or absence of time-reversal (TR) and spin-rotation (SR) symmetry. The corresponding symmetric spaces are characterized by the multiplicities of the ordinary and long roots<sup>12</sup> denoted  $m_0$  and  $m_l$  (see, e.g., Table 1 of Ref. 13). In addition to the two symmetry parameters, the degeneracy  $d$  of transmission and reflection eigenvalues has to be taken into account. Besides having definite values for certain symmetries,  $m_0$  and  $m_l$  may also be considered as interpolation parameters between different symmetry classes. In the absence of electron-electron interactions, phase randomization and local maximum entropy principles imply in a quasi-one-dimensional geometry a scaling equation, the Dorokhov-Mello-Pereyra-Kumar (DMPK) equation.<sup>14</sup> The DMPK equation yields the evolution of  $P(g)$  as a function of dimensionless length  $s$ . We set  $s = L/N\ell$ , where  $L$  is the length of the wire,  $N$  is the number of channels, and  $\ell$  is the elastic mean free path. In a wire geometry, the DMPK equation has also been generalized for the BdG structures<sup>15</sup> and for the so-called chiral classes.<sup>16</sup>

Near metallic regime the most important mechanisms inducing deviations from Gaussian  $P(g)$  are weak localization (WL) and weak antilocalization (WAL). WL and WAL result from the interference of the closed trajectories of the electron and their time-reversed conjugates. Constructive interference (WL) leads to a decrease of  $g$  while destructive interference (WAL) enhances conductance. In a conductor with length shorter or comparable with the coherence length, in the absence of magnetic field, destructive interference may be realized in a material with strong spin-orbit scattering whereas spin-rotation symmetry leads to constructive interference. The W(A)L corrections to conductivity have been used to study experimentally the quantum coherence in mesoscopic structures (see, e.g., Ref. 17 for recent mea-

surements). It is very difficult to evaluate the exact influence of W(A)L on the values of  $\langle\langle g^n \rangle\rangle$  with  $n > 3$  by such field theoretic approaches as diagrammatic techniques<sup>4,18</sup> or by the nonlinear sigma model.<sup>19</sup>

A nonperturbative treatment of the first and second cumulants based on Fourier analysis on the supersymmetric manifold was given in Ref. 20 for WD classes. The behavior of the lowest conductance cumulants  $\langle\langle g^{1,2} \rangle\rangle$  has been intensively computed for the WD ensembles ( $m_l = 1$ ) recently, especially in the metal-insulator crossover region (see, e.g., Refs. 21 and 22). In Ref. 4, based on a  $2 + \epsilon$  expansion, the expression

$$\langle\langle g^n \rangle\rangle \sim \langle g \rangle^{2-n}, \quad n < 1/s \quad (1)$$

was presented for the standard unitary ensemble ( $m_0 = 1, m_l = 1$ ). Likewise, for the symmetry class AI, but for a quasi-one-dimensional conductor (where one has  $\langle g \rangle \approx 1/s$ ), the same equation was put forward in Ref. 23. This formula has been widely accepted until now.

For the BdG wires near metallic region, the dependence of  $\langle g \rangle$  on  $m_0$  and  $m_l$  may be found in Ref. 15. Leading terms of  $\langle\langle g^2 \rangle\rangle$  (universal conductance fluctuations) and some correction terms for  $\langle\langle g^2 \rangle\rangle$  and for  $C_2$  may be found in Ref. 24. An essential singularity has been shown to occur for the first and second cumulants in the small  $s$  expansion in the chiral unitary class.<sup>25</sup> Macêdo considered in Ref. 26 the cases with  $m_0 = 2$  and  $0 \leq m_l \leq 2$  and found that the third cumulant contains only a component which is nonanalytic in  $s$  at  $s = 0$ . Localization in superconducting wires has been discussed in Refs. 15 and 27. The conductance cumulants with  $n \leq 4$  were computed in Ref. 21 for the WD classes A and AI by a Monte Carlo method.

In this paper we present a recursion equation unifying the description of the WD and BdG symmetry classes and yielding the cumulants of the order  $n < 1/s$ . For the two BdG classes with TR symmetry we find that the higher-order cumulants ( $n \geq 3$ ) contain no contributions that are analytic in  $s$  at  $s = 0$ . We elucidate the dependence of the higher cumulants on the universality class and give values for  $\langle\langle g^n \rangle\rangle$  in terms of  $m_0$  and  $m_l$ . We emphasize that even though Eq. (1) is correct for  $n = 1, 2$ , for  $2 < n < 1/s$  there exists a more appropriate expression, our Eq. (12). Furthermore, we calculate the weak-localization corrections to the current cumulants  $C_j$  with  $j \leq 10$ . We consider a disordered quantum wire, i.e., a quasi-one-dimensional geometry. For the BdG universality classes we study heat transport whereas for the WD classes our results apply also for electrical transport. We calculate the cumulants at zero temperature, zero frequency, and at low voltage in the limit  $N \rightarrow \infty, L/\ell \rightarrow \infty, s = \text{constant}$ , near the diffusive region, where one has  $1/N \ll s \ll 1$ .

## II. CUMULANTS

### A. Method

The starting point for our analysis is the generalized DMPK equation which reads<sup>14,15</sup>

$$\partial_s w_s(\boldsymbol{\lambda}) = \frac{2N}{m_0 N + 1 + m_l - m_0} \sum_{i=1}^N \frac{\partial}{\partial \lambda_i} \left\{ \times [\lambda_i(1 + \lambda_i)]^{(m_l+1)/2} J_{m_0}(\boldsymbol{\lambda}) \frac{\partial}{\partial \lambda_i} [\lambda_i(1 + \lambda_i)]^{(1-m_l)/2} \frac{w_s(\boldsymbol{\lambda})}{J_{m_0}(\boldsymbol{\lambda})} \right\}. \quad (2)$$

The variables  $\{\lambda_i\}_{i=1}^N$  are related to the transmission probabilities  $\{\tau_i\}_{i=1}^N$  of the channels  $\{i\}_{i=1}^N$  through  $\tau_i = (1 + \lambda_i)^{-1}$  while  $w_s(\boldsymbol{\lambda})$  is the distribution function for  $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_N)$ . The Jacobian  $J_{m_0}(\boldsymbol{\lambda})$  is given by  $J_{m_0}(\boldsymbol{\lambda}) = \prod_{i < j} |\lambda_i - \lambda_j|^{m_0}$ . In a long wire, for the symmetry classes D(III), additional terms enter the DMPK equation.<sup>27</sup> Near diffusive regime such components, however, are irrelevant and we will ignore them.

In order to calculate the cumulants we adopt a method reminiscent to the moment expansion method introduced in Refs. 28 and 29. It is convenient to introduce the moment generating function  $Z_s(\mathbf{q})$  and the cumulant generating function (CGF)  $\varphi_s(\mathbf{q})$ ,

$$\ln Z_s(\mathbf{q}) = \ln \langle \exp(-\mathbf{q} \cdot \mathbf{T}) \rangle_s = \varphi_s(\mathbf{q}) \quad (3)$$

in order to systematically solve the DMPK equation. Here one has  $\mathbf{T} = (T_1, \dots, T_N)$ ,  $T_k = \sum_{i=1}^N \tau_i^k$ . The expectation value  $\langle \dots \rangle_s$  is taken with respect to the distribution of  $\tau_i$ s. The conductance cumulants may be obtained from

$$\langle\langle g^n \rangle\rangle \equiv (-d)^n \left. \frac{\partial^n \varphi_s(\mathbf{q})}{\partial q_1^n} \right|_{\mathbf{q}=0}. \quad (4)$$

For large  $N$  it is natural to seek the CGF in the form of the expansion<sup>30</sup>

$$\varphi_s(\mathbf{q}) = \sum_{j=-\infty}^1 \varphi_s^{(j)}(\mathbf{q}) N^j. \quad (5)$$

In Ref. 30 it was found that in such an expansion, for the symmetry class A,  $\varphi_s^{(j)}(\mathbf{q})$  is an odd (even) polynomial of degree  $2 - j$  in  $\mathbf{q}$  with  $j$  odd (even). For all the WD and BdG classes, in the region where  $\varphi_s^{(j)}(\mathbf{q})$  may be expanded in non-negative powers of  $L/\ell$ , it may be shown by induction from Eq. (A2) in Appendix A and from the definition of the CGF, Eq. (3), that  $\varphi_s^{(j)}(\mathbf{q})$  is of the form

$$\varphi_s^{(j)}(\mathbf{q}) = \sum_{n=1}^{2-j} \sum_{k_1, \dots, k_n=1}^{\infty} A_{k_1, \dots, k_n}^{(j)}(L/\ell) \prod_{i=1}^n q_{k_i}. \quad (6)$$

We seek  $A_{k_1, \dots, k_n}^{(j)}(L/\ell)$  in the form of a rational function of  $L/\ell$ . By using Eqs. (3) and (A2) it can be shown by induction that the further condition  $1/N \ll s$  implies that  $A_{k_1, \dots, k_n}^{(j)}(L/\ell)$  takes the form

$$A_{k_1, \dots, k_n}^{(j)}(L/\ell) = a_{k_1, \dots, k_n}^{(j)}(L/\ell)^{-j} + \mathcal{O}[(L/\ell)^{-j-1}]. \quad (7)$$

Thus in the limit  $N \rightarrow \infty$ ,  $L/\ell \rightarrow \infty$ ,  $s = \text{constant}$  we obtain for all the BdG and WD classes the expansion

$$\varphi_s(\mathbf{q}) = \sum_{j=-\infty}^1 \sum_{n=1}^{2-j} \sum_{k_1, \dots, k_n=1}^{\infty} a_{k_1, \dots, k_n}^{(j)} s^{-j} \prod_i^n q_{k_i}. \quad (8)$$

This is essentially an expansion in the inverse powers of

large bare conductance and it is expected to converge in the metallic regime  $1/N \ll s \ll 1$ . We will show that, in the presence of the TR symmetry,  $a_{k_1, \dots, k_n}^{(j)}$  with  $n > 2$  actually vanish for the BdG classes CI and DIII. Thus for the classes CI and DIII, the cumulants higher than the second contain no components that are analytic in  $s$  at  $s = 0$ .

From Eq. (A2) in Appendix A one obtains the first of the coefficients  $a_1^{(1)} = -1$ . For the other coefficients we obtain after a lengthy calculation an algebraic recursive equation (see Appendix A for more details),

$$\begin{aligned} \left( -m + 2 \sum_{p=1}^n k_p \right) a_{k_1, \dots, k_n}^{(m)} &= \left[ \sum_{j=m}^1 \sum_{n_1=\max(1, n+m-j)}^{\min(2-j, n)} \sum_{p_1 \neq p_2 \neq \dots \neq p_n=1}^n k_{p_1} n_1 (n - n_1 + 1) \frac{1}{n!} \right. \\ &\times \left( \sum_{l=0}^{k_{p_1}-1} \Delta_{n_1, n, l, k_{p_1}}^{j, m} a_{k_{p_1}-l, k_{p_2}, \dots, k_{p_{n_1}}}^{(j)} a_{l+1, k_{p_{n_1}+1}, \dots, k_{p_n}}^{(m-j+1)} - \sum_{l=0}^{k_{p_1}-2} a_{k_{p_1}-l-1, k_{p_2}, \dots, k_{p_{n_1}}}^{(j)} a_{l+1, k_{p_{n_1}+1}, \dots, k_{p_n}}^{(m-j+1)} \right) \Big] \\ &+ \sum_{p=1}^n k_p \left\{ \left( \theta_1(n+1) \sum_{l=0}^{k_p-1} a_{k_p-l, l+1, k_1, \dots, k_{p-1}, k_{p+1}, \dots, k_n}^{(m+1)} + \frac{2\theta_2}{m_0 n} \sum_{r=1, r \neq p}^n k_r a_{k_p+k_r+1, k_1, \dots, k_{p-1}, k_{p+1}, \dots, k_{r-1}, k_{r+1}, \dots, k_n}^{(m+1)} \right) \right. \\ &\left. - (k_p \rightarrow k_p - 1) + \frac{\theta_3}{m_0} [(m_0 - 2)k_p + m_l - 1] a_{k_p+1, k_1, \dots, k_{p-1}, k_{p+1}, \dots, k_n}^{(m+1)} - \frac{\theta_3}{m_0} [(m_0 - 2)k_p + 2m_l - m_0] a_{k_1, \dots, k_n}^{(m+1)} \right\}. \quad (9) \end{aligned}$$

Here we have

$$\begin{aligned} a_{k_{p_1}-l, k_{p_2}, \dots, k_{p_{n_1}}}^{(j)} &= a_{k_{p_1}-l}^{(j)} \quad \text{for } n_1 = 1, \\ a_{l+1, k_{p_{n_1}+1}, \dots, k_{p_n}}^{(m-j+1)} &= a_{l+1}^{(m-j+1)} \quad \text{for } n_1 = n. \end{aligned}$$

Moreover, we denote

$$\begin{aligned} \Delta_{n_1, n, l, k_{p_1}}^{j, m} &= (1 - \delta_{j, m} \delta_{n_1, n} \delta_{l, 0})(1 - \delta_{j, 1} \delta_{n_1, 1} \delta_{l, k_{p_1}-1}), \\ \theta_1 &= \theta(-n - m), \quad \theta_2 = \theta(n - 2)\theta(2 - n - m), \\ \theta_3 &= \theta(1 - n - m) \end{aligned}$$

with

$$\theta(n) = \begin{cases} 1 & \text{for } n \geq 0, \\ 0 & \text{otherwise} \end{cases}.$$

Note also that  $\sum_{l=0}^{k_{p_1}-2} \equiv 0$  with  $k_{p_1} = 1$ . For the higher cumulants this recursion relation is conveniently evaluated by a computer. Because of Eq. (8), the coefficients  $a_{k_1, \dots, k_n}^{(m)}$  are invariant under the change of subindices. The coefficients are evaluated (1) in the order of decreasing  $m = 1, 0, -1, \dots$ , (2) for a given  $m$  in the order of

increasing  $n$ , and (3) for given  $m$  and  $n$  they may be calculated, e.g., in the order of increasing subindices.

Equation (4) implies

$$\langle\langle g^n \rangle\rangle = (-d)^n n! \sum_{m=-\infty}^1 \underbrace{a_{1, \dots, 1}^{(m)}}_n s^{-m}. \quad (10)$$

It then follows that, instead of

$$\langle\langle g^n \rangle\rangle \sim s^{n+\delta_{m_0, 2-2}}, \quad 2 < n < 1/s, \quad (11)$$

that generalizes Eq. (1) to the remaining six symmetry classes, we obtain the scaling

$$\langle\langle g^n \rangle\rangle \sim s^{n+\delta_{m_0, 2-1}}, \quad 2 < n < 1/s, \quad (12)$$

that is down by one power of  $s$  relative to the result of Altshuler, Kravtsov, and Lerner.

Calculations based on a nonperturbative microscopic approach<sup>23</sup> suggest that there exist additional nonperturbative contributions that become significant for  $n > 1/s$  and which lead to log-normal tails of the conductance distribution but are not describable within the one-parameter scaling approach. Since we cannot produce

such nonperturbative components by our method we restrict our considerations to conductance cumulants of the order  $n < 1/s$ .

References 4 and 23 missed the cancellation of the terms of the order  $\mathcal{O}(s^{n-2})$  since the numerical prefactors corresponding to  $a_{k_1, \dots, k_n}^{(2-n)}$  were not evaluated. The cancellation of the leading contribution of Eq. (1) has already been pointed out for  $n = 3$  in Refs. 30, 31, and 18. This cancellation is specific to quasi-one-dimensional geometry.<sup>18</sup> For higher  $n$  the cancellation of the leading terms has been proved in Appendix B.

## B. Conductance cumulants

For the numerical values of  $\langle\langle g^n \rangle\rangle$ , the cases  $n = 1, 2$  have been covered, e.g., in Ref. 31 for the WD classes and in Ref. 15 ( $n = 1$ ) and Ref. 24 ( $n = 1, 2$ ) for the BdG classes. Our Eq. (9) provides an efficient derivation of these results. As an application of Eq. (9) we give the results for  $n \leq 6$ ,

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$$\begin{aligned}
d^{-1}\langle\langle g \rangle\rangle &= \frac{1}{s} + \frac{(m_0 - 2m_l)}{3m_0} + [(3m_0 - 8m_l)(m_0 - 2) + 4m_l(m_l - 2)] \frac{s}{45m_0^2} \\
&\quad + \{[-21m_0 + 6m_0^2 + 2m_l(-5m_0 - 4m_l + 16)](m_0 - 2) + 8m_l(m_l - 1)(m_l - 2)\} \frac{2s^2}{945m_0^3} + \mathcal{O}(s^3), \\
d^{-2}\langle\langle g^2 \rangle\rangle &= \frac{2}{15m_0} + \frac{8(m_0 - 2)s}{315m_0^2} - [(15 + 19m_0 - 69m_l)(m_0 - 2) + 32m_l(m_l - 2)] \frac{4s^2}{4725m_0^3} + \mathcal{O}(s^3), \\
d^{-3}\langle\langle g^3 \rangle\rangle &= -\frac{8(m_0 - 2)s^2}{1485m_0^3} + [(172\,635 + 200\,793m_0 - 751\,117m_l)(m_0 - 2) + 353\,792m_l(m_l - 2)] \frac{32s^3}{638\,512\,875m_0^4} \\
&\quad + \{[-291\,960 - 1\,371\,774m_0 + 554\,337m_0^2 + (2\,303\,952 - 1\,429\,012m_0 + 205\,356m_l)m_l](m_0 - 2) \\
&\quad + 286\,720m_l(m_l - 1)(m_l - 2)\} \frac{8s^4}{212\,837\,625m_0^5} + \mathcal{O}(s^5), \\
d^{-4}\langle\langle g^4 \rangle\rangle &= \frac{512(m_0 - 2)s^3}{155\,925m_0^4} - [(17\,006\,315 + 15\,370\,851m_0 - 62\,563\,249m_l)(m_0 - 2) + 29\,630\,464m_l(m_l - 2)] \\
&\quad \times \frac{32s^4}{54\,273\,594\,375m_0^5} + \mathcal{O}(s^5), \\
d^{-5}\langle\langle g^5 \rangle\rangle &= -\frac{11\,229\,952(m_0 - 2)s^4}{3\,919\,486\,725m_0^5} + [(55\,989\,824\,345 + 45\,233\,187\,091m_0 - 192\,229\,424\,511m_l)(m_0 - 2) \\
&\quad + 91\,546\,451\,968m_l(m_l - 2)] \frac{128s^5}{510\,443\,155\,096\,875m_0^6} + \mathcal{O}(s^6), \\
d^{-6}\langle\langle g^6 \rangle\rangle &= \frac{6\,045\,601\,792(m_0 - 2)s^5}{1\,747\,609\,738\,875m_0^6} + \mathcal{O}(s^6).
\end{aligned}
\tag{13}$$


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For a normal-metal wire ( $m_l = 1$ ), the term of the order  $\mathcal{O}(s^2)$  in the second cumulant and the first term of  $\langle\langle g^3 \rangle\rangle$  agree with Ref. 31. For the symmetry class A ( $m_0 = 2, m_l = 1$ ), the subleading term of  $\langle\langle g^3 \rangle\rangle$  coincides with Ref. 26.

For the BdG classes CI and DIII ( $m_0 = 2, m_l = 0$  and  $m_0 = 2, m_l = 2$ ) the coefficients of the form  $a_{k_1}^{(m)}, a_{k_1, k_2}^{(m)}$  with  $m \leq -1$  equal zero. By a similar argument as in Appendix B one may show that this implies that all the factors of the form  $a_{k_1, \dots, k_n}^{(m)}$  with  $n \geq 3$  and any  $m$  vanish.

Thus in the case of the symmetry classes CI and DIII, the cumulants higher than the second do not contain components which are analytic in  $s$  at  $s = 0$ . For these

symmetry classes, the three lowest cumulants contain a contribution that is nonanalytic at  $s = 0$ .<sup>26</sup> We consider it likely that there exists a residual effect of this nonanalytic behavior in the higher-order cumulants too so that these cumulants do not vanish, but we cannot prove this by our perturbative method. If such a residual effect exists it is special to the cumulants higher than the second in the context of classes CI and DIII that the nonperturbative components are actually the leading contributions in the diffusive regime. This absence of an analytical expression would indicate that the diagrammatic or semiclassical pictures<sup>32</sup> do not lend themselves to the interpretation of these higher-order spectral correlations

in the BdG wires with TR symmetry.

The factor  $(m_0 - 2)/m_0$  is a consequence of W(A)L. For the WD ensemble with broken TR symmetry ( $m_0 = 2$ ,  $m_l = 1$ ) we obtain only even (odd) powers of  $s$  in  $\langle\langle g^n \rangle\rangle$  with  $n$  even (odd), in agreement with Refs. 30 and 31 whereas for the symmetry classes AI, AII, C, and D all the powers are present.

### C. Current cumulants

As a second application of Eq. (9) we have calculated the current cumulants  $C_j$  familiar from the context of full counting statistics and defined<sup>33</sup> in terms of the generating function  $S(\chi) = -\ln[\sum_N P_{t_0}(N) \exp(iN\chi)]$ ,

$$C_j \equiv -(-i)^j \frac{\partial^j}{\partial \chi^j} S(\chi) \Big|_{\chi=0}. \quad (14)$$

Here  $P_{t_0}(N)$  is the probability of  $N$  electrons traversing through the sample in a time  $t_0$  while  $\chi$  is a so-called counting field. Since we have

$$\left\langle \sum_i \tau_i^j \right\rangle_s = \langle T_j \rangle_s = - \sum_{m=-\infty}^1 a_j^{(m)} s^{-m}, \quad (15)$$

the equation<sup>34</sup>

$$C_j = \frac{deVt_0}{h} \left\langle \sum_i \left[ \tau(1-\tau) \frac{d}{d\tau} \right]^{j-1} \tau \Big|_{\tau=\tau_i} \right\rangle_s \quad (16)$$

yields for the current cumulants

$$\begin{aligned} C_1 &= Q_0 + c_0 + c_1, \\ C_2 &= \frac{1}{3}Q_0 + \frac{1}{15}c_0 - \frac{3}{7}c_1, \quad C_3 = \frac{1}{15}Q_0 + \frac{1}{315}c_0 + \frac{23}{105}c_1, \\ C_4 &= -\frac{1}{105}Q_0 - \frac{11}{1575}c_0 - \frac{401}{3465}c_1, \\ C_5 &= -\frac{1}{105}Q_0 - \frac{1}{1485}c_0 + \frac{18}{315} \frac{101}{315}c_1, \\ C_6 &= \frac{1}{231}Q_0 + \frac{47}{14} \frac{221}{189} \frac{175}{175}c_0 - \frac{24}{945} \frac{433}{945}c_1, \\ C_7 &= \frac{27}{5005}Q_0 + \frac{811}{2} \frac{027}{025}c_0 + \frac{62}{4} \frac{922}{775}c_1, \\ C_8 &= -\frac{3}{715}Q_0 - \frac{1}{516} \frac{790}{891} \frac{851}{375}c_0 - \frac{664}{70} \frac{157}{509} \frac{285}{285}c_1, \\ C_9 &= -\frac{233}{36} \frac{465}{465}Q_0 - \frac{98}{206} \frac{299}{239} \frac{813}{658} \frac{625}{625}c_0 - \frac{1}{204} \frac{095}{105} \frac{799}{825}c_1, \\ C_{10} &= \frac{6823}{969} \frac{969}{969}Q_0 + \frac{23}{3} \frac{610}{781} \frac{799}{060} \frac{591}{408} \frac{125}{125}c_0 \\ &\quad + \frac{10}{3} \frac{053}{478} \frac{185}{575} \frac{861}{575} \frac{475}{475}c_1. \end{aligned} \quad (17)$$

The next correction terms are of the order  $\mathcal{O}(s^2)$ . Here we have adopted the notations

$$\begin{aligned} Q_0 &= \frac{I_0 t_0}{e}, \quad I_0 = \frac{e^2}{h} \frac{d}{s} V, \quad c_0 = \frac{(m_0 - 2m_l)deVt_0}{3m_0 h}, \\ c_1 &= \frac{[(3m_0 - 8m_l)(m_0 - 2) + 4m_l(m_l - 2)]deVt_0 s}{45m_0^2 h}. \end{aligned} \quad (18)$$

Equations (9) and (16) provide a straightforward derivation for the leading contributions of  $C_j$  since for these terms, in Eq. (9), only the sums in square brackets with  $n = m = j = n_1 = p_1 = 1$  contribute. For the leading contributions of  $C_j$ , our results agree with those in Ref. 34. It can be shown by Eq. (9) that the correction terms of certain order  $\mathcal{O}(s^{-m})$  depend similarly on  $m_0$  and  $m_l$  with all the  $j$ 's (through  $c_0, c_1, \dots$ ) and differ only by the numerical coefficients. This was proved in Ref. 35 for the terms of the order  $\mathcal{O}(s^0)$ . While the weak-localization corrections to the current cumulants have been studied before, e.g., in Ref. 36, to our knowledge, the exact numerical values for  $j > 2$  have not been reported. Except for the first two cumulants, the ratio of the numerical factors before the universal correction term  $c_0$  and before the bare first cumulant  $Q_0$  is for even cumulants of the order, but smaller than unity and for odd cumulants smaller by about a factor of ten.

### III. DISCUSSION

In conclusion, we have presented a recursion equation covering the asymptotic behavior of the higher-order mesoscopic fluctuations for seven universality classes. We evaluated the values of the conductance cumulants  $\langle\langle g^n \rangle\rangle$  with  $n \leq 6$  and the weak-localization corrections to the current cumulants  $C_j$  for  $j \leq 10$ . We discovered two qualitative results: (1) conductance cumulants of order larger than two and smaller than  $1/s$  with  $s$  the length of the wire in units of the number of channels times mean free path scale with  $s$  with one less power than expected on the basis of naive scaling analysis and (2) the same cumulants are all vanishing in an expansion in powers of  $s$  in the two BdG symmetry classes characterized by TR symmetry, the fact from which we conjecture pure non-analytic dependence on  $s$ . As far as the noninteracting DMPK model is valid,  $P(g)$  deviates from the Gaussian shape only slightly. These deviations may, however, in principle, be detected by generating a large number of uncorrelated disorder realizations by repeatedly heating and cooling the sample.<sup>37</sup>

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### APPENDIX A: DERIVATION OF THE RECURSION EQ. (9)

The DMPK equation (2) may be expressed through  $\tau_i s$  instead of  $\lambda_i s$  (for the WD ensembles, see Ref. 29).

Rewriting Eq. (A1) of Ref. 24, which is a direct consequence of the DMPK equation, in terms of  $\tau_i s$  yields for  $\exp(-\mathbf{q} \cdot \mathbf{T})$  an evolution equation

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$$\begin{aligned} \frac{(m_0 N + 1 + m_l - m_0)}{2N} \partial_s \langle \exp(-\mathbf{q} \cdot \mathbf{T}) \rangle_s^{(m_0, m_l)} = & \left\langle \left\{ \sum_{k=1}^{\infty} k q_k \frac{m_0}{2} \left[ \left( \sum_{l=0}^{k-1} T_{k-l} T_{l+1} \right) - \left( \sum_{l=0}^{k-2} T_{k-l-1} T_{l+1} \right) \right] + \right. \right. \\ & \left. \sum_{k,l=1}^{\infty} k l q_k q_l (T_{k+l} - T_{k+l+1}) + \sum_{k=1}^{\infty} k q_k \left[ \left( 1 - \frac{m_0}{2} \right) k (T_{k+1} - T_k) + \frac{(1 - m_l)}{2} T_{k+1} + \frac{(2m_l - m_0)}{2} T_k \right] \right\} \exp(-\mathbf{q} \cdot \mathbf{T}) \right\rangle_s^{(m_0, m_l)}. \end{aligned} \quad (\text{A1})$$


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For the CGF  $\varphi_s(\mathbf{q})$ , the terms of the order  $\mathcal{O}(N^{m+1})$  imply

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$$\begin{aligned} \frac{m_0}{2N} \partial_s \varphi_s^{(m)} + \frac{(1 + m_l - m_0)}{2N} \partial_s \varphi_s^{(m+1)} = & \sum_{k_1=1}^{\infty} k_1 q_{k_1} \left\{ \frac{m_0}{2} \left[ \sum_{l=0}^{k_1-1} \left( \sum_{j=m}^1 \frac{\partial \varphi_s^{(j)}}{\partial q_{k_1-l}} \frac{\partial \varphi_s^{(m-j+1)}}{\partial q_{l+1}} + \frac{\partial^2 \varphi_s^{(m+1)}}{\partial q_{k_1-l} \partial q_{l+1}} \right) \right. \right. \\ & + \sum_{k_2=1}^{\infty} k_2 q_{k_2} \frac{\partial \varphi_s^{(m+1)}}{\partial q_{k_1+k_2+1}} \left. \right] - \frac{m_0}{2} [k_1 \rightarrow k_1 - 1] + \frac{(m_0 - 2)}{2} k_1 \left( \frac{\partial \varphi_s^{(m+1)}}{\partial q_{k+1}} - \frac{\partial \varphi_s^{(m+1)}}{\partial q_k} \right) + \frac{(m_l - 1)}{2} \frac{\partial \varphi_s^{(m+1)}}{\partial q_{k_1+1}} \\ & \left. + \frac{(m_0 - 2m_l)}{2} \frac{\partial \varphi_s^{(m+1)}}{\partial q_{k_1}} \right\}. \end{aligned} \quad (\text{A2})$$


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Making use of the expression for  $\varphi_s^{(m)}(\mathbf{q})$  given by Eqs. (5) and (8) and considering the terms of the order  $\mathcal{O}(s^{-m} \prod_{i=1}^n q_{k_i})$  we arrive, after a lengthy algebra, at an algebraic recursive equation, our Eq. (9).

### APPENDIX B: CANCELLATION OF THE TERMS $\mathcal{O}(s^{n-2})$ FOR $\langle\langle g^n \rangle\rangle$ WITH $n \geq 3$

Let us consider Eq. (9) with a factor of the form  $a_{q_1, q_2, \dots, q_n}^{(2-n)}$  on the left-hand side (lhs). On the right-hand side (rhs) of Eq. (9), only the sums in square brackets and the  $r$  sum on the third line may then contribute. There are no factors of the form  $a_{q_1, q_2, \dots, q_n}^{(3-n)}$  in Eq. (8). The factors  $a_{q_1, q_2, q_3}^{(-1)}$  vanish for all the WD classes.<sup>31</sup> Equation (9) implies  $a_{q_1, q_2, q_3}^{(-1)} = 0$  for all the symmetry classes since the terms containing  $m_l$  cannot contribute to  $a_{q_1, q_2, q_3}^{(-1)}$ . It is

thus sufficient to show that on the rhs, the joint contribution of (i) sums containing products of factors of the form  $a_{q_1}^{(1)}$  and  $a_{q_1, q_2, \dots, q_n}^{(2-n)}$ , (ii) sums containing products of factors  $a_{q_2, q_3}^{(0)}$  and  $a_{q_1, q_2, \dots, q_{n-1}}^{(3-n)}$ , and (iii) factors of the form  $a_{q_1, q_2, \dots, q_{n-1}}^{(3-n)}$  vanishes.

For the induction argument, let us consider Eq. (9) with  $m = 2 - n_0$ ,  $n = n_0$  fixed and assume that we have  $a_{q_1, q_2, \dots, q_{n_0-1}}^{(3-n_0)} = 0$  for all positive integer values of  $q_i$ . The induction assumption implies that the contributions of the forms (ii) and (iii) vanish such that we have  $a_{q_1, q_2, \dots, q_{n_0}}^{(2-n_0)} = 0$  with  $q_1 = q_2 = \dots = q_{n_0} = 1$ . This implies that also the contribution (i) vanishes and we have  $a_{q_1, q_2, \dots, q_{n_0}}^{(2-n_0)} = 0$  with arbitrary  $q_i$ s. The induction assumption holds with  $n_0 = 4$  since we have  $a_{q_1, q_2, q_3}^{(-1)} = 0$ . Thus we obtain  $a_{q_1, q_2, \dots, q_n}^{(2-n)} = 0$  for arbitrary  $n \geq 3$ .

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